Journal of Engineering Mathematics **26**: 267–280, 1992.

Havelock wavemakers, Westergaard dams and the Rayleigh hypothesis

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Received 5 November 1990; accepted 12 January 1991

Abstract. Water of constant finite depth fills a semi-infinite channel, with a wavemaker, W, at one end. The generation of small-amplitude gravity waves by harmonic oscillations of W leads to a linear boundary-value problem for a velocity potential, ϕ. For vertical, plane wavemakers, there is a theory due to Havelock in which ϕ is represented as a convergent series of eigenfunctions, with coefficients determined by the boundary condition on W. We show that the same representation (with different coefficients) can also be used for some wavemakers with other shapes; the allowable geometries and forcings are determined. This is a hydrodynamic analogue of the so-called Rayleigh hypothesis in the theory of gratings. Similar results obtain for the hydrodynamic loading of dams due to short-duration earthquakes.

1. Introduction

The method of separation of variables is often used to solve boundary-value problems in rectangular regions. For semi-infinite regions, given in Cartesian coordinates (x, y) by $x \geq 0$ and $0 \leq y \leq h$, it usually leads to eigenfunction expansions, where each eigenfunction satisfies the governing partial differential equation (herein taken to be Laplace's equation), homogeneous boundary conditions on $y = 0$ and $y = h$, and an appropriate condition as $x \to \infty$; the coefficients in the expansion are determined by the remaining boundary condition on $x = 0$.

Two examples are of interest here, namely Havelock wavemakers and Westergaard dams. In both of these, the semi-infinite region is filled with incompressible, inviscid fluid, $y = h$ is the bottom and $y = 0$ is the mean free surface. Small oscillations of a vertical wavemaker at $x = 0$ will generate surface waves that radiate towards $x = +\infty$; the theory for this problem was given by Havelock [6]. Short-duration earthquakes will induce hydrodynamic pressures on the vertical face of a dam at $x = 0$; the theory for this problem (with a flat free surface, and also for compressible fluids) was given by Westergaard [31]. Both theories are described in text-books; see, e.g. Dean and Dalrymple [3, §6.3] for Havelock's theory, and Newmark and Rosenblueth [18, §6.2] for Westergaard's theory.

Suppose, now, that the end-wall (i.e. the wavemaker or the dam face) is not plane and vertical, so that the fluid region is no longer rectangular. We still have our set of eigenfunctions, each one of which satisfies all conditions of the boundary-value problem, save one: can they be combined so as to satisfy the boundary condition on the end-wall? In other words, can the solution of the boundary-value problem be represented *everywhere in the fluid*, as a convergent series of 'rectangular' eigenfunctions with coefficients determined by the end-wall boundary condition? In general, we have a negative answer. However, for *some* geometries and for *some* forcings, the representation is valid. In this paper, we give a method for determining the allowable geometries and forcings. This method is an adaptation of a method due to van den Berg and Fokkema [30] for determining the limitations of the so-called *Rayleigh hypothesis* in the theory of gratings.

Non-vertical wavemakers have recently attracted some attention. Raichlen and Lee [22] considered the generation of waves by a flat, inclined wavemaker, defined by

$$x = \alpha y . \tag{1.1}$$

They used an integral-equation method, involving a simple Green's function and an approximate radiation condition. The same geometry has been considered by other authors, using rectangular eigenfunctions [10, 15, 32]. Other geometries have been considered using rectangular eigenfunctions [33], null-field methods [13, 21] and perturbation methods [10, 21].

Non-vertical dams are of great practical importance. The effects of surface waves are usually neglected, leading to a Dirichlet condition on the free surface. For a flat inclined face, given by (1.1), there is an exact solution to the boundary-value problem, given by Chwang [2]. Rectangular eigenfunctions have been used to treat other geometries [1, 34].

All the methods mentioned above that use expansions in terms of appropriate rectangular eigenfunctions are discussed further in Section 9.

2. Formulation of the problems

Consider a semi-infinite channel, D, filled with water. The water has constant finite depth h. At one end, there is a wavemaker W. We suppose that W makes small time-harmonic oscillations, and are required to determine the amplitude of the waves radiated to infinity. We assume that the water is incompressible and inviscid, and that the motion is irrotational and two-dimensional. Hence, there exists a velocity potential $\mathrm{Re}\{\phi(x, y)\, \mathrm{e}^{-\mathrm{i}\omega t}\}$, where ω is the radian frequency of oscillation, and (x, y) are Cartesian coordinates. We choose the latter so that the mean free surface and the horizontal bottom occupy portions of the lines $y = 0$ and $y = h$, respectively. We specify the shape of the wavemaker according to

$$W = \{(x, y) \colon x = w(y), 0 \leq y \leq h\} ,$$

and take the fluid domain as

$$D = \{(x, y) \colon x > w(y), 0 < y < h\} .$$

Finally, we locate the origin so that

$$0 \leq w(y) \leq c \quad \text{for } 0 \leq y \leq h ,$$

whence the wavemaker is bounded by the two vertical lines $x = 0$ and $x = c$.

We now formulate a boundary-value problem for ϕ.

Wavemaker problem

Determine $\phi(x, y)$ so that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi(x, y) = 0 \quad \text{in the water } D ; \tag{2.1}$$

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{on the bottom } y = h,\ x > w(h)\,; \tag{2.2}$$

$$K\phi + \frac{\partial \phi}{\partial y} = 0 \quad \text{on the free surface, } y = 0,\ x > w(0)\,; \tag{2.3}$$

$$\frac{\partial \phi}{\partial n} = U \quad \text{on the wavemaker, } W\,; \tag{2.4}$$

and

$$\phi(x, 0) \sim \frac{-\mathrm{i}g}{\omega} A\, \mathrm{e}^{\mathrm{i}k_0 x} \quad \text{as } x \to +\infty\,. \tag{2.5}$$

Here, $K = \omega^2/g$, where g is the acceleration due to gravity; $U(y)$ is the prescribed normal velocity on W; $\partial/\partial n$ denotes normal differentiation at a point on W in the direction from W into D; k_0 is the unique positive real root of the dispersion relation,

$$K = k_0 \tanh k_0 h\,; \tag{2.6}$$

and A is a complex constant whose magnitude gives the amplitude of the waves radiated to infinity.

We assume that the Wavemaker Problem has at most one solution. This uniqueness assumption is known to be valid if

$$w(y) \leq w(0) + y \tan \beta_m \quad \text{for } 0 \leq y \leq h\,, \tag{2.7}$$

where $\beta_m \simeq 44\frac{1}{3}°$ [27], but may also be valid for a wider class of geometries.

We shall also consider the following special case of the Wavemaker Problem, obtained by formally letting $K \to \infty$.

Dam problem

Determine $\phi(x, y)$ so that ϕ satisfies (2.1), (2.2), (2.4),

$$\phi(x, 0) = 0 \quad \text{for } x > w(0) \tag{2.8}$$

and

$$\phi \to 0 \quad \text{as } x \to +\infty\,. \tag{2.9}$$

This problem arises when a rigid dam, with face W undergoes a constant horizontal acceleration in the x-direction, as the result of an earthquake. If the earthquake has a short duration, gravitational effects are negligible, i.e. the hydrodynamic pressure induced on the dam can be calculated by assuming that the free surface is flat. The physical quantity of most interest here is the pressure distribution on the dam; this is proportional to $\phi(w(y), y)$.

3. Havelock wavemaker theory

Havelock's classical theory [6, 29] gives the exact solution of the Wavemaker Problem when the wavemaker W is vertical, i.e. when $w(y) = 0$ for $0 \leq y \leq h$. Separation of variables leads to a set of *Havelock wavemaker functions*, $\{\Phi_m(x, y)\}$ where

$$\Phi_0(x, y) = e^{ik_0x} Y_0(y) , \qquad \Phi_n(x, y) = e^{-k_nx} Y_n(y) ,$$
$$Y_0(y) = C_0 \cosh k_0(h - y) , \qquad Y_n(y) = C_n \cos k_n(h - y) ,$$
$$C_0 = 2(2k_0h + \sinh 2k_0h)^{-1/2} , \qquad C_n = 2(2k_nh + \sin 2k_nh)^{-1/2} ,$$

k_n are the positive real roots of

$$K + k_n \tan k_nh = 0 , \tag{3.1}$$

and $n = 1, 2, \ldots$. The functions $\{\Phi_m\}$ satisfy (2.1), (2.2), (2.3) and (2.5). The functions $\{Y_m(y)\}$ $(m = 0, 1, 2, \ldots)$ are complete and orthogonal over $0 \leq y \leq h$; the constants C_m were inserted so that

$$\int_0^h Y_m(y) Y_n(y)\, dy = \frac{1}{k_m} \delta_{mn} , \tag{3.2}$$

where δ_{ij} is the Kronecker delta.

For the vertical wavemaker at $x = 0$, we can write

$$\phi(x, y) = ib_0\Phi_0(x, y) + \sum_{n=1}^{\infty} b_n\Phi_n(x, y) .$$

Applying the boundary condition (2.4), by differentiating term by term, and then using the orthogonality relation (3.2), we obtain

$$b_m = -\int_0^h U(y) Y_m(y)\, dy ,$$

for $m = 0, 1, 2, \ldots$. In particular, the wave amplitude at infinity is

$$|A| = (\omega/g)|b_0| C_0 \cosh k_0h .$$

4. Westergaard's solution for a vertical dam

Letting $Kh \to \infty$, we see that $k_0h \to \infty$, $Y_0(y) \to 0$ for $y > 0$,

$$k_nh \to (2n - 1)\frac{\pi}{2} = \lambda_nh , \tag{4.1}$$

say, and

$$Y_n(y) \to \sqrt{\frac{2}{\lambda_nh}}\, (-1)^{n+1} \sin \lambda_ny .$$

Now, consider a vertical dam at $x = 0$, and look for a solution in the form

$$\phi(x, y) = \sum_{n=1}^{\infty} b_n\, e^{-\lambda_nx} \sin \lambda_ny .$$

Applying the boundary condition (2.4) gives

$$U(y) = -\sum_{n=1}^{\infty} \lambda_n b_n \sin \lambda_n y\,, \quad 0 < y < h\,. \tag{4.2}$$

The orthogonality of $\{\sin \lambda_n y\}$ over $0 < y < h$ then gives

$$b_n = -\frac{2}{\lambda_n h} \int_0^h U(y) \sin \lambda_n y \, \mathrm{d}y\,.$$

For example, if the dam moves horizontally as a rigid body, we have

$$U(y) = \alpha g/\omega\,,$$

where α is a dimensionless constant, whence

$$b_n = -\frac{2\alpha g}{\omega h \lambda_n^2}\,.$$

This result was obtained by Westergaard [31].

We note here that (4.2) is a Fourier quarter-range sine series: the function $U(y)$ is first extended into $h < y < 2h$ using $U(y) = U(2h - y)$, and then into $-2h < y < 0$ using $U(y) = -U(-y)$. This extended function will be discontinuous at $y = 0$ (unless $U(0) = 0$), whence Gibb's phenomenon will be present. This feature is quite clear in numerical experiments; see Figure 11 in [25].

5. The Rayleigh hypothesis

Suppose, now, that the wavemaker W is not vertical. Can we still write

$$\phi(x, y) = \mathrm{i} b_0 \Phi_0(x, y) + \sum_{n=1}^{\infty} b_n \Phi_n(x, y)\,, \tag{5.1}$$

where the series is uniformly convergent for all points $(x, y) \in D \cup W$? If so, we can apply the boundary condition (2.4) by differentiating term by term, and then try to determine the coefficients b_n $(n = 0, 1, 2, \ldots)$, perhaps by collocation ('point-matching') or a Galerkin method; see Section 9 and [20, §1.2.8].

We call the assumption that (5.1) is a valid representation for ϕ in $D \cup W$ the *Rayleigh hypothesis*, as Rayleigh [23, §272a], [24], made a similar assumption in his work on acoustic scattering by a grating (an infinite, periodic corrugated surface). The Rayleigh hypothesis has generated a large literature since it was first questioned by Lippmann [11]; for a review, see [16].

It is known that the Rayleigh hypothesis is valid for some, but not all, geometries. Conditions for its validity have been devised by Hill and Celli [7], van den Berg and Fokkema [30], DeSanto [4], Schlup [26], Maystre and Cadilhac [14] and Millar [17]. In this paper, we show that the method of van den Berg and Fokkema can be adapted to the Wavemaker Problem and to the Dam Problem.

6. Expansion beyond $x = c$

It is clear that we can expand $\phi(x, y)$ as (5.1) for $x \geqslant c$: one merely imagines that there is a vertical wavemaker at $x = c$, with a certain (unknown) variation of $\partial\phi/\partial x$ over $x = c$. Indeed, this observation is behind the so-called 'equivalent wavemaker method', where one tries to make plausible choices for $\partial\phi/\partial x$ [5].

So, suppose that we write

$$\phi(x, y) = \mathrm{i}\tilde{b}_0(x, y) + \sum_{n=1}^{\infty} \tilde{b}_n \Phi_n(x, y)\,, \tag{6.1}$$

for $x \geqslant c$. Then the uniqueness theorem for the Wavemaker Problem implies that

$$\tilde{b}_n = b_n\,, \quad n = 0, 1, 2, \ldots \tag{6.2}$$

This is not a trivial result, for we can obtain explicit formulae for the coefficients in (6.1), in terms of the boundary values of ϕ and $\partial\phi/\partial n$ on W, whereas the expansion (5.1) may not converge on W.

We begin with the fundamental solution (Green's function) G, defined by

$$\begin{aligned} G(x, y; \xi, \eta) = {} & \frac{1}{2} \log \frac{(x-\xi)^2 + (y-\eta)^2}{(x-\xi)^2 + (y+\eta)^2} \\ & - 2 \int_0^{\infty} \frac{\cosh k(h-y) \cosh k(h-\eta)}{\cosh kh(k \sinh kh - K \cosh kh)} \cos k(x-\xi)\, \mathrm{d}k \\ & - 2 \int_0^{\infty} \mathrm{e}^{-kh} \frac{\sinh ky \sinh k\eta}{k \cosh kh} \cos k(x-\xi)\, \mathrm{d}k\,, \end{aligned}$$

where the path of integration passes below the pole of the integrand at $k = k_0$ [9]. $G(x, y; \xi, \eta)$ is the potential at (x, y) due to a simple wave source at (ξ, η) in an infinite channel, $\{-\infty < x < \infty, 0 < y < h\}$. G has the bilinear expansion [9, 13]

$$G(x, y; \xi, \eta) = -\pi \sum_{m=0}^{\infty} \alpha_m(\xi, \eta) \Phi_m(x, y) \tag{6.3}$$

for $0 < \xi < x$, where

$$\alpha_0(\xi, \eta) = \mathrm{i}\, \mathrm{e}^{-\mathrm{i}k_0\xi} Y_0(\eta)\,, \qquad \alpha_n(\xi, \eta) = \mathrm{e}^{k_n\xi} Y_n(\eta)$$

and $n = 1, 2, \ldots$. An application of Green's theorem in D to ϕ and G gives

$$2\pi\phi(P) = \int_W \left\{ \frac{\partial\phi(q)}{\partial n} G(P, q) - \phi(q) \frac{\partial}{\partial n_q} G(P, q) \right\} \mathrm{d}s_q\,, \tag{6.4}$$

where $P = (x, y)$ is a point in D (not on W), $q = (\xi, \eta)$ is a point on W and $\partial/\partial n_q$ denotes normal differentiation at q. If we restrict P so that $x > c$, we can use (6.3) in (6.4); (6.1), (6.2) and (2.4) then give

$$\mathrm{i}b_0 = \frac{1}{2} \int_W \left(\phi \frac{\partial\alpha_0}{\partial n} - U\alpha_0 \right) \mathrm{d}s \tag{6.5}$$

and

$$b_m = \frac{1}{2} \int_W \left(\phi \frac{\partial \alpha_m}{\partial n} - U\alpha_m \right) \mathrm{d}s \tag{6.6}$$

for $m = 1, 2, \ldots$.

7. Application of the method of van den Berg and Fokkema

We now determine sufficient conditions for the uniform convergence of (5.1) in the region $x \geq 0$ (this region contains $D \cup W$). First, it is convenient to extend the Wavemaker Problem by reflection across the bottom, so that $\phi(x, y)$ and $w(y)$ are symmetric about $y = h$. This leads to a boundary-value problem in the extended domain

$$D_e = \{(x, y): x > w(y), 0 < y < 2h\},$$

with $w(0) = w(2h)$. We also assume here that $w'(y)$ is continuous for $0 \leq y \leq 2h$, except possibly for some isolated points on $x = 0$; as we shall see, the Rayleigh hypothesis is definitely false if this condition is not met.

By the 'root test', the series (5.1) will be uniformly convergent for $x \geq 0$, $0 \leq y \leq h$, if

$$\limsup_{n \to \infty} |b_n \Phi_n(0, y)|^{1/n} < 1. \tag{7.1}$$

For large n, $k_n h = n\pi + O(1/n)$ whence $C_n = O(n^{-1/2})$ and so (7.1) reduces to

$$\limsup_{n \to \infty} |b_n|^{1/n} < 1. \tag{7.2}$$

If this holds, we can differentiate (5.1) term by term and apply the boundary condition (2.4) to give

$$\sum_{n=0}^{\infty} b_n \Psi_n(y) = f(y), \quad 0 < y < 2h, \tag{7.3}$$

where

$$f(y) = U(y)\sqrt{1 + (w'(y))^2},$$

$$\Psi_n(y) = -k_n C_n \, \mathrm{e}^{-k_n w(y)} \{\cos k_n(h - y) - w'(y) \sin k_n(h - y)\}$$

for $n = 1, 2, \ldots$, and Ψ_0 is defined similarly.

Next, we determine the behaviour of b_n for large n, so that we can test (7.2). We do this by extending (7.3) to *complex* values of y. Let us extend into the *lower* half-plane, $\mathrm{Im}(y) \leq 0$. In this half-plane,

$$\lim_{n \to \infty} |\Psi_n(y)|^{1/n} = |\zeta(y)|, \tag{7.4}$$

where

$$\zeta(y) = \exp\left\{ \frac{\pi}{h} (\mathrm{i}y - w(y)) \right\} \tag{7.5}$$

and we have used the estimate $k_n h \sim n\pi$ for large n. So, instead of (7.3), it is natural to consider the *power series*

$$\sum_{n=0}^{\infty} b_n \zeta^n = F(\zeta), \tag{7.6}$$

say, in the complex ζ-plane. The radius of convergence of this series is R, where

$$R^{-1} = \limsup_{n\to\infty} |b_n|^{1/n} < 1, \tag{7.7}$$

if (7.2) holds, i.e. we need $R > 1$.

Note that we could have extended (7.3) into the *upper* half-plane, $\mathrm{Im}(y) \geq 0$. This would lead to the replacement of ζ in (7.4) and (7.6) with $\tilde{\zeta}$, say, where

$$\tilde{\zeta}(y) = \exp\left\{-\frac{\pi}{h}(\mathrm{i}y + w(y))\right\}.$$

However, since $w(y)$ is real for real y, its analytic continuations into the complex y-plane satisfy $w(\bar{y}) = \bar{w}(y)$. Hence, $|\tilde{\zeta}| = |\zeta|$, and so both power series have the same radius of convergence. Thus, it is sufficient to consider only $\mathrm{Im}(y) \leq 0$.

Now, the formula (7.5) defines a mapping from the strip

$$S = \{0 \leq \mathrm{Re}(y) \leq 2h, \mathrm{Im}(y) \leq 0\} \tag{7.8}$$

into the ζ-plane. This mapping is conformal except where $\zeta' = 0$ or $\zeta' = \infty$, i.e. where

$$\mathrm{i} - w'(y) = 0 \tag{7.9}$$

or at singularities of $w(y)$. The image in the ζ-plane of the line $\{0 \leq \mathrm{Re}(y) \leq 2h, \mathrm{Im}(y) = 0\}$ (these values of y correspond to the extended wavemaker) is a closed curve C, symmetric about $\mathrm{Im}(\zeta) = 0$. On this curve, $|\zeta| \leq 1$, whence C is strictly contained inside C_R, the circle of convergence of (7.6), if (7.2) holds.

C_R passes through the singularity of the power series (7.6) that is closest to $\zeta = 0$, at $\zeta(y_0)$, say. Thus, $y = y_0$ is either a singularity of $w(y)$, or a singularity of $f(y)$ or a zero of $\zeta'(y)$; the latter are given by (7.9) as

$$\mathrm{i} - w'(y_0) = 0, \qquad \mathrm{Im}(y_0) \leq 0. \tag{7.10}$$

Then, $R = |\zeta(y_0)|$ and so (7.7) gives

$$\limsup_{n\to\infty} |b_n|^{1/n} = |\zeta(y_0)|^{-1},$$

which describes the behaviour of b_n for large n, if the Rayleigh hypothesis is valid; this will be the case if (7.7) holds, i.e. if

$$\mathrm{Re}\{\mathrm{i}y_0 - w(y_0)\} > 0,$$

where y_0 solves (7.10).

Rather than dealing with inequalities, it is convenient to consider a family of wavemakers, given by

$$w(y) = c\tilde{w}(y) ,$$

say, where $|\tilde{w}(y)| \leq 1$. Suppose that $c_{\max}$ is the smallest value of c for which

$$\mathrm{Re}\{\mathrm{i}y_0 - c\tilde{w}(y_0)\} = 0 . \tag{7.11}$$

Then it can be shown ([30], p. 30) that the Rayleigh hypothesis holds for $0 \leq c < c_{\max}$ and fails for $c \geq c_{\max}$.

7.1. Summary

Suppose that $w(y)$ and $f(y)$ are regular in the strip S, defined by (7.8); this is usually the case. Then, there are two equations to be solved, namely (7.10) and (7.11):

$$\mathrm{i} - c\tilde{w}'(y_0) = 0 , \tag{7.12}$$

$$\mathrm{Re}\{\mathrm{i}y_0 - c\tilde{w}(y_0)\} = 0 . \tag{7.13}$$

These are to be solved in the strip S for c and y_0; since they are linear in c, eliminate c and solve for y_0. For each y_0, determine c and set $c_{\max}$ to be the smallest of these values. Then, the Rayleigh hypothesis is valid for $0 \leq c < c_{\max}$, and the series (5.1) will converge everywhere in the fluid and on the wavemaker itself.

8. Examples

In this section, we apply the foregoing theory to some particular wavemaker geometries. In all cases, $w(y)$ is regular. If W is subjected to a rigid-body motion, for example, then $f(y)$ is regular too.

EXAMPLE 1. Suppose that W is given by

$$x = w(y) = \tfrac{1}{2}c(1 + \cos \mu_l y) , \tag{8.1}$$

where $c \geq 0$, $\mu_l = l\pi/h$ and l is a positive integer. For example, if $l = 1$, this wavemaker is one half of one period of a cosine curve, meeting the free surface perpendicularly at $(c, 0)$ and meeting the bottom perpendicularly at $(0, h)$.

Equations (7.12) and (7.13) become

$$\mathrm{i} + \tfrac{1}{2}c\mu_l \sin \mu_l y_0 = 0 \quad \text{and} \quad \mathrm{Re}\{\mathrm{i}y_0 - \tfrac{1}{2}c - \tfrac{1}{2}c \cos \mu_l y_0\} = 0 .$$

Set

$$\mu_l y_0 = -\mathrm{i}Y + 2m\pi , \quad m = 0, 1, \ldots, l ,$$

where Y is real and positive; this gives all the singularities in the strip S. We obtain

$$2 = c\mu_l \sinh Y \quad \text{and} \quad 2Y = c\mu_l(\cosh Y + 1)\,.$$

Eliminating c gives

$$Y \sinh Y = \cosh Y + 1\,.$$

This has one positive root at

$$Y \simeq 1.5434\,, \tag{8.2}$$

whence

$$\frac{c_{\max}}{h} \simeq \frac{0.2850}{l}\,, \quad l = 1, 2, \ldots \tag{8.3}$$

For $l = 1$, this condition is the same as for acoustic scattering by a grating of the form (8.1); see, e.g. [16]. In this case, $w(y)$ has period $2h$. For higher (integer) values of l, $w(y)$ has period $2h/l$. In acoustics, one would use eigenfunctions with the same periodicity (in y) as the geometry, because the acoustic field also has this property. This leads to

$$\frac{c_{\max}}{h/l} \simeq 0.2850\,,$$

i.e. to (8.3) again. For water waves, one uses eigenfunctions determined by the depth of water (the potential is not periodic in y!), leading to (8.3).

EXAMPLE 2. Suppose that we change the sign in (8.1) to give

$$w(y) = \tfrac{1}{2}c(1 - \cos \mu_l y)$$

with c and μ_l as for Example 1. The singularities are now given by

$$\mu_l y_0 = -\mathrm{i}Y + (2m + 1)\pi\,, \quad m = 0, 1, \ldots, l - 1\,,$$

where, as before, Y is given by (8.2) and $c_{\max}$ is given by (8.3).

EXAMPLE 3. Let us return to Example 1, but now allow l to be real, with $0 < l < 1$. Thus, we consider a wavemaker W given by

$$w(y) = c\,\frac{\cos \mu_l y - \cos l\pi}{1 - \cos l\pi}\,. \tag{8.4}$$

As before, W meets the free surface perpendicularly at $(c, 0)$; it meets the bottom at $(0, h)$, but not perpendicularly: $w'(h) = -\mu_l c \cot(l\pi/2) < 0$. Note that $w'(y)$, as defined for $h < y < 2h$ by reflection and then for all y by periodicity, is continuous for all y except at $y = (2n + 1)h$ ($n = 0, \pm 1, \pm 2, \ldots$), where $x = 0$. This means that C has a corner at $\zeta = -1$, but is otherwise smooth and strictly contained within $|\zeta| = 1$. Hence, the series (5.1) will be convergent everywhere on W, except at the single point $(x, y) = (0, h)$, provided there is no other singularity inside $|\zeta| = 1$ (otherwise C_R would be smaller than $|\zeta| = 1$). The location of this singularity depends on l. It is on $|\zeta| = 1$ if c and $\mu_l y_0 = -\mathrm{i}Y_l$ satisfy

$$(1 - \cos l\pi) = c\mu_l \sinh Y_l \quad \text{and} \quad (1 - \cos l\pi)Y_l = c\mu_l(\cosh Y_l - \cos l\pi)\,.$$

Eliminating c gives

$$Y_l \sinh Y_l = \cosh Y_l - \cos l\pi\,.$$

For example, if $l = \frac{1}{2}$, we find that $Y_l \simeq 1.1997$ and

$$\frac{c_{\max}}{h} \simeq 0.4219 \quad (l = \tfrac{1}{2})\,. \tag{8.5}$$

EXAMPLE 4. Extending Example 2 to real values of l, with $0 < l < 1$, we consider

$$w(y) = c\,\frac{1 - \cos \mu_l y}{1 - \cos l\pi}\,.$$

W meets the free surface perpendicularly at the origin; it meets the bottom at (c, h), but not perpendicularly. As in Example 3, there is a discontinuity in $w'(y)$ at $y = h$, but now it occurs where $x \neq 0$. This leads to an entirely different situation. For C touches $|\zeta| = 1$ at $\zeta = +1$, and has a corner at $\zeta = -\exp\{-c\pi/h\} = \zeta_c$, say. Since points on C satisfy $|\zeta_c| \leq |\zeta| \leq 1$, and the series (7.6) diverges for $|\zeta| = |\zeta_c|$, we see that the series (5.1) diverges *everywhere* on W.

EXAMPLE 5. Consider the parabolic form for W given by

$$w(y) = cy(2h - y)/h^2\,. \tag{8.6}$$

W meets the bottom perpendicularly at (c, h). For this geometry, we can find y_0 explicitly. It is given by

$$y_0 = h - \tfrac{1}{2}\mathrm{i}h^2/c\,,$$

whence

$$\frac{c_{\max}}{h} = \frac{1}{2}\,. \tag{8.7}$$

This result was also obtained by DeSanto [4, Example 3] for acoustic scattering by a grating. Note that (8.6) and (8.4) with $l = \frac{1}{2}$ are similar curves, and so we expect the numerical values for $c_{\max}/h$, given by (8.7) and (8.5), respectively, to be similar.

EXAMPLE 6. For the quartic form,

$$w(y) = cy^2(2h - y)^2/h^4\,,$$

W meets both the bottom and the free surface perpendicularly. Again, we can find y_0 explicitly:

$$y_0 = h(1 - \mathrm{i}3^{-1/2})$$

exactly, whence

$$\frac{c_{\max}}{h} = \frac{3\sqrt{3}}{16} \simeq 0.3248 .$$

9. Numerical methods

Suppose that the Rayleigh hypothesis is valid for a given W and U. Then, we can expect that a simple method, such as collocation, could be used to compute b_n in (5.1). Thus, with

$$E_N(y) = \sum_{n=0}^{N} b_n(N)\Psi_n(y) - f(y)$$

(the coefficients $b_n(N)$ will vary with N), solve

$$E_N(y_n) = 0 , \quad n = 0, 1, 2, \ldots , N ,$$

where $(w(y_n), y_n)$, $n = 0, 1, 2, \ldots , N$, are points on W.

Suppose, now, that the Rayleigh hypothesis is not valid. Then, a simple collocation method is unlikely to succeed: typically, $E_N(y)$ will not be small when $y \neq y_n$. In this situation, other numerical schemes have been tried. Wu [32, 33] has collocated at M points, with $M > N$, leading to an overdetermined system, which he solved in a least-squares sense: in [32], he used $N = 5$ and $M = 20$ for an inclined flat wavemaker; in [33], he used $N = 15$ and $M = 200$ for a heaving wedge; in both cases, he presented results for the wave amplitude at infinity. Kachoyan and McKee [10] have used a Galerkin scheme, in which they solved

$$\int_0^h E_N(y)X_n(y)\,\mathrm{d}y = 0 , \quad n = 0, 1, 2, \ldots , N ; \tag{9.1}$$

they chose

$$X_n(y) = Y_n(y) ,$$

the vertical eigenfunctions defined in Section 3. They considered the inclined flat wavemaker, given by

$$w(y) = \alpha y$$

(with $\alpha = c/h$), and found that their method did not converge for $\alpha > 1$. Later, McKee [15] obtained results up to $\alpha = 1.25$ by using a Shanks transform.

A different choice for X_n in (9.1) is

$$X_n(y) = \{1 + (w'(y))^2\}^{-1/2}\bar{\Psi}_n(y) ,$$

where the overbar denotes the complex conjugate. This choice is equivalent to satisfying the boundary condition (2.4) in a least-squares sense, i.e. to minimizing

$$\Omega_N = \int_W \left| \sum_{n=0}^{N} b_n(N) \frac{\partial}{\partial n_q} \Phi_n(q) - U(q) \right|^2 \mathrm{d}s_q \, .$$

This approach has been used for various dam problems by Avilés and Sánchez-Sesma [1, 25] and by Wu and Yu [34]. It has a theoretical basis; see, e.g. Millar [16] and Ikuno and Yasuura [8]. Thus, completeness of $\{(\partial/\partial n_q)\Phi_n(q)\}$ (for expanding functions defined on W, in $L_2(W)$) can be proved, using methods described in [12]. This means that there exist coefficients $b_n(N)$ such that $\Omega_N \to 0$ as $N \to \infty$. The coefficients satisfy

$$b_n(N) \to b_n \quad \text{as } N \to \infty \, ,$$

where b_n are defined by (6.5) and (6.6) and give the expansion for $\phi(x, y)$ in $x \geqslant c$; in particular, this method yields a convergent approximation to the wave amplitude at infinity (proportional to b_0). Finally, the approximation

$$\mathrm{i} b_0 \Phi_0(x, y) + \sum_{n=1}^{N} b_n(N) \Phi_n(x, y)$$

converges uniformly in all closed subsets of D. These results are independent of the Rayleigh hypothesis (they do require uniqueness for the boundary-value problem). All these aspects of the least-squares method are clearly described in Millar's well-known paper [16], although they bear repetition here. The theory requires (i) that W and its extension is smooth, and (ii) that the integration in (9.1) is performed exactly. The numerical experiments in [1, 25, 34] are for dams whose geometries violate (i); non-uniform behaviour of $E_N(y)$ was observed (see Figures 13 and 15 in [25]). For wavemakers, (ii) will be violated; guides for numerical evaluation in related problems are given in [8]. The effects of corners have also been analysed, in the context of the method of least squares for acoustic scattering by a cylinder, by Okuno and Yasuura [19]; similar considerations should be useful for the present problems.

10. Discussion and conclusions

For *some* non-vertical wavemakers, we have shown that the potential can be expanded as a convergent series of Havelock wavemaker functions, everywhere in the water and (almost) everywhere on the wavemaker itself. The allowable geometries and forcings can be determined by a simple extension of known methods for testing the Rayleigh hypothesis in acoustics.

Similar techniques should be applicable to some other related water-wave problems. For example, Ursell [28] used his multipole potentials to expand the potential everywhere outside a heaving half-immersed circular cylinder. Can the same set of multipoles be used to expand the potential everywhere outside cylinders of other cross-sections? Results similar to those described by Millar [16] are to be expected.

Finally, we observe that all of our results are applicable, unchanged, to the Dam Problem. This is because $k_n h$ and $\lambda_n h$, defined by (3.1) and (4.1), respectively, both grow like $n\pi$ as $n \to \infty$.

References

1. J. Avilés and F.J. Sánchez-Sesma, Hydrodynamic pressures on dams with nonvertical upstream face. *J. Engng. Mech.* 112 (1986) 1054–1061.
2. A.T. Chwang, Hydrodynamic pressures on sloping dams during earthquakes. Part 2. Exact theory. *J. Fluid Mech.* 87 (1978) 343–348.
3. R.G. Dean and R.A. Dalrymple, *Water Wave Mechanics for Engineers and Scientists.* Englewood Cliffs: Prentice-Hall (1984).
4. J.A. DeSanto, Scattering from a perfectly reflecting arbitrary periodic surface: an exact theory. *Radio Sci.* 16 (1981) 1315–1326.
5. M. Greenhow and M. Simon, A note on the equivalent wavemaker method. *Appl. Ocean Res.* 7 (1985) 107–112.
6. T.H. Havelock, Forced surface-waves on water. *Phil. Mag.* Series 7, 8 (1929) 569–576.
7. N.R. Hill and V. Celli, Limits of convergence of the Rayleigh method for surface scattering. *Phys. Rev. B* 17 (1978) 2478–2481.
8. H. Ikuno and K. Yasuura, Improved point-matching method with application to scattering from a periodic surface. *IEEE Trans.* AP-21 (1973) 657–662.
9. F. John, On the motion of floating bodies II. *Comm. Pure Appl. Math.* 3 (1950) 45–101.
10. P.J. Kachoyan and W.D. McKee, Wave forces on steeply-sloping sea walls. *J. Engng. Math.* 19 (1985) 351–362.
11. B.A. Lippmann, Note on the theory of gratings. *J. Opt. Soc. Amer.* 43 (1953) 408.
12. P.A. Martin, On the null-field equations for water-wave scattering problems. *IMA J. Appl. Math.* 33 (1984) 55–69.
13. P.A. Martin, On the T-matrix for water-wave scattering problems. *Wave Motion* 7 (1985) 177–193.
14. D. Maystre and M. Cadilhac, Singularities of the continuation of fields and validity of Rayleigh's hypothesis. *J. Math. Phys.* 26 (1985) 2201–2204.
15. W.D. McKee, Wave forces on steeply-sloping sea walls: oblique incidence. *J. Engng. Math.* 22 (1987) 87–99.
16. R.F. Millar, The Rayleigh hypothesis and a related least-squares solution to scattering problems for periodic surfaces and other scatterers. *Radio Sci.* 8 (1973) 785–796.
17. R.F. Millar, Singularities and the Rayleigh hypothesis for solutions to the Helmholtz equation. *IMA J. Appl. Math.* 37 (1986) 155–171.
18. N.M. Newmark and E. Rosenblueth, *Fundamentals of Earthquake Engineering.* Englewood Cliffs: Prentice-Hall (1971).
19. Y. Okuno and K. Yasuura, Numerical algorithm based on the mode-matching method with a singular-smoothing procedure for analysing edge-type scattering problems. *IEEE Trans.* AP-30 (1982) 580–587.
20. R. Petit (ed.), *Electromagnetic Theory of Gratings*, Topics in Current Physics 22. Berlin: Springer-Verlag (1980).
21. D.J. Pizer, *A Generalised Wavemaker Problem*, M.Sc. Dissertation, University of Manchester (1985).
22. F. Raichlen and J.J. Lee, An inclined-plate wave generator, *Proc. 16th Coastal Engineering Conf.*, Hamburg (1978) 388–399.
23. Lord Rayleigh (J.W. Strutt), *The Theory of Sound*, 2nd edn, Vol. 2. London: Macmillan & Co. (1896).
24. Lord Rayleigh, On the dynamical theory of gratings. *Proc. Roy. Soc. A* 79 (1907) 399–416.
25. F.J. Sánchez-Sesma and J. Avilés, Authors' closure to discussion on [1]. *J. Engng. Mech.* 114 (1988) 1103–1106.
26. W.A. Schlup, On the convergence of the Rayleigh ansatz for hard-wall scattering on arbitrary periodic surface profiles. *J. Phys. A: Math. Gen.* 17 (1984) 2607–2619.
27. M.J. Simon and F. Ursell, Uniqueness in linearized two-dimensional water-wave problems. *J. Fluid. Mech.* 148 (1984) 137–154.
28. F. Ursell, On the heaving motion of a cylinder on the surface of a fluid. *Quart. J. Mech. Appl. Math.* 2 (1949) 218–231.
29. F. Ursell, R.G. Dean and Y.S. Yu, Forced small-amplitude water waves: a comparison of theory and experiment. *J. Fluid Mech.* 7 (1959) 33–52.
30. P.M. van den Berg and J.T. Fokkema, The Rayleigh hypothesis in the theory of reflection by a grating. *J. Opt. Soc. Amer.* 69 (1979) 27–31.
31. H.M. Westergaard, Water pressures on dams during earthquakes. *Trans. ASCE* 98 (1933) 418–433.
32. Y.C. Wu, Waves generated by an inclined-plate wave generator. *Int. J. Numer. Meth. in Fluids* 8 (1988) 803–811.
33. Y.C. Wu, Plunger-type wavemaker theory. *J. Hydraulic Res.* 26 (1988) 483–491.
34. Y.C. Wu and D.J. Yu, Trefftz method for hydrodynamic pressure on rigid dams with non-vertical upstream face. *Int. J. Numer. Meth. in Fluids* 9 (1989) 1–7.